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BOUNDARY LAYERS NEAR THE PLANE FREE BOUNDARY OF A FLUID CAUSED BY AXISYMMETRICAL TANGENTIAL STRESSES†

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Spatial steady-state flows of an incompressible fluid in the boundary layers near a free surface with specified surface tangential stresses are investigated. Under conditions of axial symmetry and assuming a plane free boundary, self-similar solutions of the three-dimensional equations of the boundary layer are constructed numerically. The asymptotic solutions for small and large values of the load are obtained.

1. CONSIDER the non-linear steady-state problem of the flow of an incompressible fluid in an unbounded region D under the action of a system of tangential stresses $T(x, y, z)$ specified at the a priori unknown free boundary Γ, for the system of Navier–Stokes equations, with low viscosity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g}, \quad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

$$\begin{gathered} p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} + \sigma(k_1 + k_2) + p_*, \quad (x,y,z)\in\Gamma \\ 2\nu\rho[\Pi\mathbf{n} - (\mathbf{n}\Pi\mathbf{n})\mathbf{n}] = \mathbf{T}, \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad (x,y,z)\in\Gamma \end{gathered} \tag{1.2}$$

Here $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z) : \mathbf{g} = g\mathbf{e}_z$, $\mathbf{e}_z = (0, 0, 1)$ is the unit vector of the z axis, $\mathbf{g}$ is the acceleration due to gravity, ρ is the fluid density, $\mathbf{n}$ is the unit vector of the outward normal to Γ, Π is the deformation rate tensor, k_1 and k_2 are the principal curvatures of the surface Γ (a priori unknown), σ is the surface tension, and p_* and $T = (T_1, T_2)$ are specified normal and tangential stresses at the free surface Γ, the latter may be caused, for example, by wind. The velocity field is assumed to vanish at infinity.

We know the empirical equation [1]

$$\mathbf{T} = c_D\rho_a\mathbf{U}_a|\mathbf{U}_a| \tag{1.3}$$

where ρ_a is the density of air, c_D is an empirical coefficient of the order of 2×10^{-3}, and U_a is the air velocity in the air flow at a distance from the fluid (water) surface. Formula (1.3) is obtained from the well-known Prandtl formula for the tangential stresses for free turbulence $T = \rho c\delta|U_a - U_m|dU|dy = \rho c|U_a - U_m|(U_a - U_m)$, where δ is the thickness of the boundary layer in the air, and U_m is the air velocity at the free surface of the fluid.

Taking into account the fact that $U_m \leqslant U_a$ we obtain the quadratic dependence of the friction on the velocity U_a (1.3) (we note that $U_m/U_a = O(10^{-2})$ [1]).

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In an ideal fluid there are no surface tangential stresses, so that the boundary conditions for T are not specified. When the fluid viscosity is small, the boundary layer is formed near the free surface. This layer compensates for the discrepancy caused by the tangential stresses. We note the results in [2–4] of a study of the equations of the steady boundary layer near the free surface for specified surface tangential stresses caused by heating of the boundary ($\mathbf{T} = \nabla_\Gamma \sigma$, $\nabla_\Gamma = \nabla - (\mathbf{n}\cdot\nabla)\mathbf{n}$ is the gradient along the surface Γ, and σ is the known varying surface tension). The solvability of the plane problem of the extension of the boundary layer near Γ was proved in [3].

We will reduce problem (1.1), (1.2) to dimensionless form. Let L and T_* be the characteristic scales of the length and tangential stress. The dimensionless pressure p' is defined by the relation $p = T_* p' - \rho g z$. We will denote the characteristic velocity scale by U. Changing to dimensionless variables in the Navier–Stokes equations and multiplying them by LU^{-2} we find that the coefficient of the Laplace operator equals $\nu/(UL)$. We will denote it by ε^2. Making the stretching transformation in the boundary layer and equating the orders of magnitude of the viscous and inertial terms we find that the thickness of the boundary layer is of the order of ε. Next we change to dimensionless parameters in the boundary conditions for the tangential stresses and take into account the stretching transformation. Equating the orders of the principal terms in these conditions we obtain the relation $T_* = \rho\nu U/(\varepsilon L)$. From this we find the characteristic scale of the velocity in the boundary layer $U = (LT_*^2\rho^{-2}\nu^{-1})^{1/3}$. We now represent the small parameter ε in the form $\varepsilon = (\rho\nu^2 L^{-2} T_*^{-1})^{1/3}$. Small ε corresponds to a small value of the coefficient of viscosity. We note that the relation $\varepsilon \sim \nu^{2/3} T_*^{-1/3}$ has been obtained [2] for the Marangoni boundary layers (where we should set $T_* = A|\sigma_t|$), A is the scale of the temperature gradient, and σ_t is the known coefficient in the relation $\sigma = \sigma_0 - |\sigma_t|(\theta - \theta_0)$, θ is the temperature).

The asymptotic expansions of the solution of problem (1.1), (1.2) as $\varepsilon \to 0$ are constructed in the form

$$\begin{gathered}\mathbf{v} \sim \mathbf{h}_0 + \varepsilon(\mathbf{h}_1 + \mathbf{v}_1) + \ldots \\ p' \sim q_0 + \varepsilon(q_1 + p_1) + \ldots, \quad \zeta \sim \zeta_0 + \varepsilon\zeta_1 + \ldots\end{gathered} \tag{1.4}$$

Here $z = \zeta(x, y)$ is the equation of the free boundary, which is a priori unknown.

Let D_Γ be the region of the boundary layer. Then $\mathbf{h}_k$, q_k will be functions of the type representing the solutions of the boundary-layer problem in the region D_Γ. The functions $\mathbf{v}_1$ and p_1 which describe non-viscous flow and satisfy Euler's equations, will determine the solution of the problem outside the region D_Γ.

2. Let us formulate the problem for the principal terms $\mathbf{h}_0$ and q_0 of the asymptotic form (1.4) which determine the flow in the boundary-layer region. We introduce the local orthogonal coordinates ξ, φ, θ, near the surface Γ, where ξ is the distance between the point $N(x, y, z)$ and the surface Γ, and φ, θ are the curvilinear coordinates on Γ of the reference point of the normal dropped from the point N. Using the terminology of [5] we apply the second (internal) iterative process to (1.1) and (1.2). Let $h_{\varphi k}$, $h_{\theta k}$, $h_{\xi k}$ be the components of the vector $\mathbf{h}_k$ in the local coordinates. We substitute (1.4) into (1.1) and (1.2), introduce the stretching transformation $s = \xi/\varepsilon$ and expand $\mathbf{v}_1$ in a Taylor series in powers of ξ. Equating the coefficients of ε^{-1}, ε^0 to zero we obtain that $h_{\xi 0} = 0$, and $h_{\varphi 0}$, $h_{\theta 0}$, $h_{\xi 1}$ satisfy the dimensionless equations of the boundary layer which are identical with the Prandtl equations (note that the boundary-layer thickness is of the order of $\nu^{1/2}$ near the solid wall and $\nu^{2/3}$ near the free boundary, in this case)

$$\begin{gathered}g_\varphi^{-1} h_{\varphi 0}\frac{\partial h_{\varphi 0}}{\partial\varphi} + g_\theta^{-1} h_{\theta 0}\frac{\partial h_{\varphi 0}}{\partial\theta} + H_{\xi 1}\frac{\partial h_{\varphi 0}}{\partial s} + \delta^{-1}\frac{\partial g_\varphi}{\partial\theta} h_{\varphi 0} h_{\theta 0} - \delta^{-1}\frac{\partial g_\theta}{\partial\varphi} h_{\theta 0}^2 = \frac{\partial^2 h_{\varphi 0}}{\partial s^2} \\ g_\varphi^{-1} h_{\varphi 0}\frac{\partial h_{\theta 0}}{\partial\varphi} + g_\theta^{-1} h_{\theta 0}\frac{\partial h_{\theta 0}}{\partial\theta} + H_{\xi 1}\frac{\partial h_{\theta 0}}{\partial s} + \delta^{-1}\frac{\partial g_\theta}{\partial\varphi} h_{\varphi 0} h_{\theta 0} - \delta^{-1}\frac{\partial g_\varphi}{\partial\theta} h_{\varphi 0}^2 = \frac{\partial^2 h_{\theta 0}}{\partial s^2}\end{gathered} \tag{2.1}$$

$$\frac{\partial}{\partial\varphi}(g_\theta h_{\varphi 0})+\frac{\partial}{\partial\theta}(g_\varphi h_{\theta 0})+\frac{\partial}{\partial s}(\delta H_{\xi 1})=0;\quad H_{\xi 1}=h_{\xi 1}+\mathbf{v}_1\mathbf{n}|_\Gamma$$

Thus, we have the following boundary conditions at the free surface and at infinity

$$\partial h_{\varphi 0}/\partial s=-T_\varphi,\quad \partial h_{\theta 0}/\partial s=-T_\theta$$
$$H_{\xi 1}=0,\quad (s=0);\quad h_{\varphi 0}=h_{\theta 0}=0\quad (s=\infty)$$

Here g_φ, g_θ are the Lamé coefficients of the surface, $\delta=g_\varphi g_\theta$. The higher-order approximations correspond the linear boundary-value problems, which are not discussed here. Below only the first approximation is constructed.

The principal term in the asymptotic expansion (1.4) for the pressure is obtained in the form

$$q_0=-k_1\int_s^\infty h_{\varphi 0}^2 ds-k_2\int_s^\infty h_{\theta 0}^2 ds$$

The boundary-value problem for the functions $\mathbf{v}_1$, p_1, ζ_0 which determine the flow outside the boundary layer and the asymptotic form of the free boundary is obtained by applying the first (external) iterative process [5] to system (1.1), (1.2). The equation of the surface Γ_0 $(z=\zeta_0)$ is obtained, apart from infinitesimals of higher order, by integrating the relation in the dimensional form

$$\sigma(k_{10}+k_{20})+k_{10}\int_0^\infty h_{\varphi 0}^2 ds+k_{20}\int_0^\infty h_{\theta 0}^2 ds=\rho gz+p_* \tag{2.2}$$

where k_{10}, k_{20} are the principal curvatures of the surface Γ_0.

The non-viscous flow outside the boundary layer is determined by solving the boundary-value problem

$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1=-\nabla p_1,\quad \operatorname{div}\mathbf{v}_1=0$$
$$\mathbf{v}_1\mathbf{n}|_{\Gamma_0}=H_{\xi 1}|_{\xi=\infty},\quad \mathbf{v}_1=0\quad (x^2+y^2+z^2=\infty)$$

The system obtained allows the potential flow of a fluid. Denoting the flow potential $(\mathbf{v}_1=\nabla\Phi)$ by Φ, we obtain the Neumann problem with respect to Φ

$$\Delta\Phi=0$$
$$\partial\Phi/\partial n|_{\Gamma_0}=H_{\xi 1}|_{\xi=\infty},\quad \nabla\Phi=0\quad (x^2+y^2+z^2=\infty)$$

The results of a determination of the free boundary from formula (2.2) were discussed in [4]. We note that in the plane and axisymmetric cases Eq. (2.2) can be simplified considerably. For example, in the plane problem we assume $h_{\theta 0}=k_2=T_\theta=0$, $g_\theta=1$, integrate the first equation in (2.1) with respect to s over the semi-axis $[0,\ \infty]$, integrate by parts, take into account the boundary conditions and integrate the resulting expression with respect to φ. As a result, we obtain

$$\int_0^\infty h_{\varphi 0}^2 ds=\int_{\varphi_0}^{\varphi} T_\varphi g_\varphi d\varphi+\int_0^\infty f_0^2(s)ds \tag{2.3}$$

Here $f_0(s)=h_{\varphi 0}(s,\ \varphi_0)$ is the velocity profile in the boundary layer in the cross-section $\varphi=\varphi_0$. We note that it is convenient to choose φ_0, for which the value of f_0 is known. Substituting

(2.3) into relation (2.2) we obtain

$$k_{10}(\sigma+\int\limits_{\varphi_0}^{\varphi} T_\varphi g_\varphi d\varphi+\int\limits_0^\infty f_0^2(s)ds)=\rho gz+p_* \tag{2.4}$$

If there is no tangential load ($T_\varphi=f_0=0$), we obtain from Eq. (2.4) the well-known equation of the free boundary of an ideal fluid at rest $k_{10}\sigma=\rho gz+p_*$.

Let us determine the form of the free surface in the case when the tangential and normal stresses acting on it are specified. We write Eq. (2.4) in Cartesian coordinates

$$\frac{\zeta_0''}{(1+\zeta_0'^2)^{3/2}}\left[\sigma+\int\limits_0^x T_\varphi(1+\zeta_0'^2)^{1/2}dx+\int\limits_0^\infty f_0^2 ds\right]=\rho g\zeta_0+p_* \tag{2.5}$$

The function ζ_0 vanishes at infinity. We integrate Eq. (2.5) numerically for specified values of T_φ and p_*, for example, for symmetric loads $p_*=\exp(-x^2)$, $T_\varphi=\lambda x\exp(-x^2)$ and non-symmetric loads $p_*=-\exp[4(1-2x^2)]$, $T_\varphi=\lambda\delta(x)$ (here $\delta(x)$ is the delta-function). For $T_\varphi=0$ the free boundary is a bowl or a cusp. When the amplitude λ of the tangential load increases, the depth of the bowl and the height of the cusp decrease and tend to zero for large values of λ, i.e. when the tangential stresses smooth out the free surface.

3. Let us consider the problem of a three-dimensional boundary layer near a free surface. We will assume that $p_*=0$. Then the free boundary equation (2.2) has the solution $z=0$, $k_{10}=k_{20}=0$, i.e. if there are no surface normal stresses at the free boundary this free surface may be a plane up to $O(\varepsilon)$. Next we consider the case of the plane free surface at which the tangential load is specified, whose components are

$$T_r=-\lambda_1 r^n,\quad T_\theta=-\tau r^n$$

Here r,θ,z are cylindrical coordinates. It is convenient to choose the minus sign, so that in view of the vector formula (1.3) the components T_r and T_θ have the same direction as the corresponding components of the velocity vector $\mathbf{U}_a$ and can have any sign. The exponential load can be caused by the wind with velocity distribution $|\mathbf{U}_a|=c\varphi^{n/2}$ in (1.3).

Assuming the fluid flow to be axisymmetric we set $\mathbf{h}_0=\mathbf{h}_0(z/\varepsilon,\ r)$. The system of boundary layer equations (2.1) in cylindrical coordinates $\varphi=r$, $g_\varphi=1$, $g_\theta=r$ has the self-similar solution

$$\begin{aligned} h_{r0}&=r^{(2n+1)/3}F_1(s_1),\quad h_{\theta0}=r^{(2n+1)/3}G_1(s_1)\\ H_{z1}&=r^{(n-1)/3}H_1(s_1),\quad s_1=zr^{(n-1)/3}/\varepsilon \end{aligned} \tag{3.1}$$

Taking into account the fact that the functions (3.1) are independent of the coordinate θ, we find that system (2.1) is invariant under the replacement of the functions $h_{\varphi0}$, $h_{\theta0}$, $H_{\xi1}$, T_φ, T_θ by $h_{\varphi0}$, $-h_{\theta0}$, $H_{\xi1}$, T_φ, $-T_\theta$. Thus we restrict ourselves to the case when $\tau\geqslant0$. We make the change of variables

$$\eta=\tau^{1/3}s_1,\quad G=\tau^{2/3}G_1,\quad F=\tau^{2/3}F_1,\quad H=\tau^{1/3}H_1,\quad \lambda=\lambda_1\tau^{-1}$$

In order to determine the functions F,G and H we substitute (3.1) into Eq. (2.1) and obtain the boundary-value problem (the prime denotes a derivative with respect to η)

$$\begin{aligned} &F''+G^2-\frac{2n+1}{3}F^2-HF'+\frac{1-n}{3}\eta FF'=0\\ &G''-HG'-\frac{2n+4}{3}FG'+\frac{1-n}{3}\eta FG'=0 \end{aligned} \tag{3.2}$$

$$H' + \frac{2n+4}{3}F + \frac{n-1}{3}\eta F' = 0$$
$$G'(0) = 1, \quad F'(0) = \lambda, \quad H(0) = F(\infty) = G(\infty) = 0$$

For $n = 1$, problem (3.2) is distinguished from the well-known problem for the boundary layer near a disc rotating about an axis perpendicular to its plane by the boundary conditions (the fluid far from the disc is at rest). In the disc problem, the functions $F(0)$ and $G(0)$ are specified at $s = 0$, while in problem (3.2) the derivatives $F'(0)$, $G'(0)$ are specified.

System (3.2) was integrated numerically for various values of n and λ. For $\lambda = 0$ the tangential load simulates a tornado, and for $\lambda \neq 0$ the radial tangential stresses are superimposed on the tornado. The parameter λ takes positive as well as negative values depending on whether the load radial component is directed towards or away from the centre of the tornado.

Figure 1 shows the dependence of the functions $F(0)$ and $G(0)$ (proportional to the radial and peripheral components of the velocity at the free boundary) on the parameter n. Curves 1 and 2 are plots of the functions $F(0)$ and $-G(0)$ for $\lambda = 0$. When the parameter n increases the magnitudes of the radial and peripheral components of the velocity decrease.

In Fig. 2 curves 1–3 show the dependence of the functions $F(0)$, $G(0)$ and $F(0)/G(0)$ on λ for $n = 2$. Both the functions $F(0)$ and $G(0)$ decrease monotonically when λ increases and tend to $-\infty$ as $\lambda \to +\infty$. The ratio $F(0)/G(0)$ determines the tangent of the angle between the velocity vector at the free boundary and the peripheral direction. This angle varies from $-90°$ ($\lambda = -\infty$) up to $23.06°$ as $\lambda \to +\infty$. Unlike the disc problem, all the components of the velocity (3.1) depend monotonically on the s-coordinate for $n = 1$, $n = 2$ and $\lambda \leqslant 0$. A non-monotonic dependence of F on s is only found for positive values of λ.

We quote the asymptotic solutions of system (3.2) for $\lambda \to -\infty$. We introduce the new variable $t = (-\lambda)^{1/3}\eta$ and represent the functions F, G and H in the form of expansions

$$F = (-\lambda)^{2/3} f_0(t) + \ldots, \quad G = (-\lambda)^{-1/3} g_0(t) + \ldots, \quad H = (-\lambda)^{1/3} h_0(t) + \ldots$$

The principal terms f_0, g_0, h_0 are found by solving the boundary-value problem

$$f_0'' = \frac{2n+1}{3}f_0'^2 + h_0 f_0' + \frac{n-1}{3}tf_0 f_0'$$
$$g_0'' = \frac{2n+4}{3}f_0 g_0 + h_0 g_0' + \frac{n-1}{3}tf_0 g_0; \quad h_0' = -\frac{2n+4}{3}f_0 + \frac{1-n}{3}tf_0' \tag{3.3}$$
$$f_0'(0) = -1, \quad g_0'(0) = 1, \quad h_0(0) = 0, \quad f_0(\infty) = g_0(\infty) = 0$$

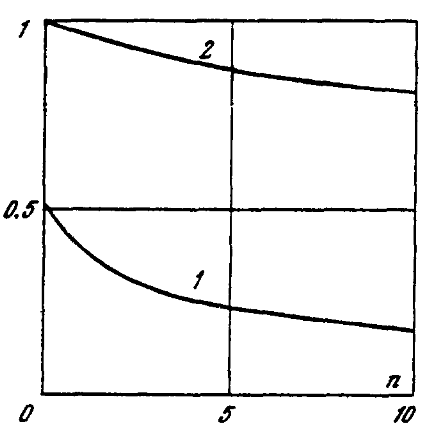

FIG. 1.

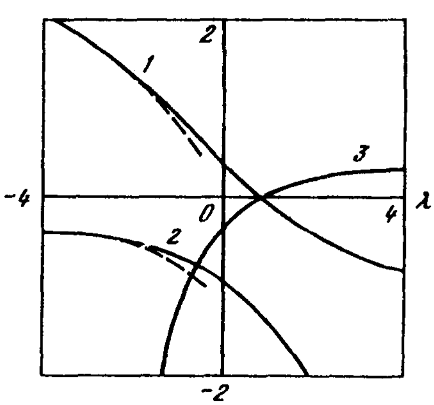

FIG. 2.

Calculations show that $f_0(0)=1.0564$, $g_0(0)=-0.7763$ and $h_0(\infty)=-1.4705$ for $n=0$. When n increases, the values of $f_0(0)$ and $|g_0(0)|$ decrease monotonically. The function $f_0(t)$ decreases monotonically, and $g_0(t)$ increases as t increases on the semi-axis $[0, \infty)$ for fixed n. We will give formulae for calculating the velocity at the free boundary for $n=2$

$$F(0)\sim 0.8052(-\lambda)^{2/3}+\ldots,\quad G(0)\sim -0.6833(-\lambda)^{-1/3}+\ldots\quad (\lambda\to-\infty)$$

The dashed lines in Fig. 2 show the asymptotic solution as $\lambda\to-\infty$.

The asymptotic solution of system (3.2) for $\lambda\to+\infty$ is constructed from the formulae

$$F\sim\lambda^{2/3}f(\eta_1)+\ldots,\quad G\sim\lambda^{2/3}g(\eta_1)+\ldots,\quad H\sim\lambda^{1/3}h(\eta_1)+\ldots,\quad \eta_1=\lambda^{1/3}\eta$$

The functions f, g and h are found from the boundary-value problem which only differs from (3.3) by the extra term $-g^2$ added to the right-hand side of the first equation and the boundary conditions for the derivatives replaced by $f'(0)=1$ and $g'(0)=0$.

For $n=2$ we quote the numerical values $f(0)=-0.4768$, $g(0)=-1.1202$. The function $f(0)$ increases monotonically as the parameter n increases and reaches a value of -0.3396 at $n=10$, and the function $g(0)$ falls monotonically to -1.3510 at $n=10$. On the semi-axis $[0, \infty)$ the function $g(\eta_1)$ increases monotonically while f and h have one extremum each. We note that the functions f, g, h define the solution of the problem for the three-dimensional boundary layer near the free boundary when the peripheral component T_θ of the tangential load vanishes and its radial component is directed towards the centre of the tornado $T_\varphi<0$. In the case of $T_\varphi>0$ the boundary layer is two dimensional: $g=h_{\theta 0}=0$ (the system has a solution if $f'(0)=-1$).

We will consider the three-dimensional flow, caused by the tangential load, in the boundary layer near a plane free boundary in the case when, unlike the problem considered above, the fluid rotates at infinity as a solid, with angular velocity ω: $\upsilon_{r0}=\upsilon_{z0}=0$, $\upsilon_{\theta 0}=\omega r$, $p_0=\frac{1}{2}\omega^2r^2+c(z\to-\infty)$. The velocities in the boundary layer and in the external flow are of the same order, so the asymptotic expansions, unlike (1.4), should be constructed in the form

$$\mathbf{v}\sim\mathbf{h}_0+\mathbf{v}_0+\varepsilon(\mathbf{h}_1+\mathbf{v}_1)+\ldots$$
$$p\sim q_0+p_0+\varepsilon(q_1+p_1)+\ldots,\quad \zeta=O(\varepsilon)$$

Let us construct the self-similar solution, assuming that the vector $\mathbf{h}_0$ is independent of the θ coordinate. The components of the vector $\mathbf{w}_0=\mathbf{h}_0+\mathbf{v}_0$ to satisfy Eqs (1.2), with the extra term $-dp_0/dr$ added to the right-hand side of the first equation. Assume the tangential load at the free boundary in the form $T_r=0$, $T_\theta=\tau r$ (this load may be caused by wind with a velocity field $\mathbf{U}_a$ having the components $U_{ar}=0$, $U_{a\theta}=\pm\sqrt{(|\tau|r/(c_D\rho_a))}$ according to (1.3)). We will represent the solution of the problem in the form

$$w_{r0}=rF(s),\quad w_{\theta 0}=rG(s),\quad w_{z0}=0,\quad w_{z1}=H(s),\quad s=z/\varepsilon$$

The functions F, G and H are found by solving the boundary-value problem

$$\begin{aligned}&F''-F^2-HF'+G^2-\omega^2=0\\&G''-HG'-2FG=0,\quad H'+2F=0\\&G'(0)=\tau,\quad F'(0)=H(0)=F(\infty)=0,\quad G(\infty)=\omega\end{aligned}\tag{3.4}$$

The system obtained was integrated for $\omega=1$ and various values of τ. The results of the calculation are shown in Fig. 3. Curves 1–3 are plots of the functions F, $G_1=G-\omega$ and $H/2$ against s for $\tau=1$, and curves 4–6 for $\tau=-1$. Note that the parameter τ may be positive as well

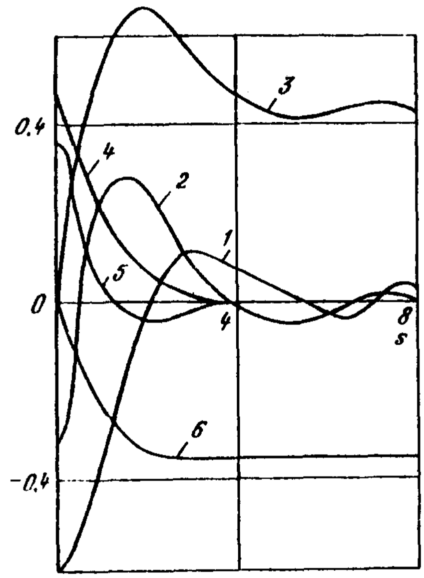

FIG. 3.

as negative, and that for $T_r = 0$ the direction of the peripheral component of the air velocity U_∞ is determined by the vector formula (1.3) Unlike the case of a fluid which is at rest at infinity, the components of the velocity vector vary non-monotonically with distance from the free boundary. The boundary layer thickness increases with ω. When $\Pi p|\tau| \ll 1$ system (3.4) can be linearized, and its asymptotic solution is constructed by explicit formulae

$$F(s) = -\tau\sqrt{2/\omega}\sin(s\sqrt{\omega} + \pi/4)\exp(-s\sqrt{\omega}) + O(\tau^2)$$

$$G(s) = \omega + \tau\sqrt{2/\omega}\sin(s\sqrt{\omega} - \pi/4)\exp(-s\sqrt{\omega}) + O(\tau^2) \quad (\tau \to 0)$$

We will give the principal terms of the asymptotic form of system (3.4) for high values of the angular velocity and $\tau = O(1)$

$$F = -\tau^{2/3}\Omega^{-1/2}\cos(\eta\sqrt{\Omega/2} - \pi/4)\exp(-\eta\sqrt{\Omega/2}) + O(1/\Omega)$$
$$\Omega = 2\omega\tau^{-2/3}, \quad \eta = s\tau^{1/3} \quad (\Omega \to \infty) \tag{3.5}$$

The expression for G is obtained from (3.5) by replacing cos by −sin.

System (3.4) was integrated numerically for $\tau = 0$ and a specified radial component of the tangential load $F'(0) = \lambda$. As in the case considered, the velocity vector oscillates as the distance from the boundary increases. For small values of λ the asymptotic form of the solution is obtained from (3.5) by replacing F, η, $\tau^{2/3}$ and Ω by $-F$, s, λ and 2, respectively.

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